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# The decay of massive closed superstrings with maximum angular momentum

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Abstract: We study the decay of a very massive closed superstring (i.e. $\alpha' M^2 \gg 1$) in the unique state of maximum angular momentum. This is done in flat ten-dimensional spacetime and in the regime of weak string coupling, where the dominant decay channel is into two states of masses $M_1,\ M_2$. We find that the lifetime surprisingly grows with the first power of the mass $M$: $\mathcal{T} = c\alpha'\ M$. We also compute the decay rate for each values of $M_1,\ M_2$. We find that, for large $M$, the dynamics selects only special channels of decay: modulo processes which are exponentially suppressed, for every decay into a state of given mass $M_1$, the mass $M_2$ of the other state is uniquely determined.

Keywords: Superstrings and Heterotic Strings, Conformal Field Models in String Theory.

## Contents



## 1. Introduction

In type-II superstring theory in flat, ten-dimensional non-compact spacetime, all massive strings are generally expected to be unstable quantum mechanically by a decay into lighter particles. Massive strings are the key ingredient of string theory and crucial for the consistency of the theory. Despite many years of study of string theory, very little is known about the way a massive string decays.

Earlier calculations of decay properties are in [1]–[6], and more recent studies can be found in [7, 8, 9]. A calculation for states with maximum angular momentum in the open string theory was given in [6]. An inclusive decay rate was computed in [7]. A more recent calculation for open and closed superstring theory for *generic* states (which have angular momentum much less than the maximum value) is done in [9].

For small string coupling, the dominant elementary process is the decay into two particles. If these particles are massive, then each of them will subsequently decay into two ligther particles, and the process ends when only massless particles remain.

Here we shall explicitly compute the rate for the decay of the massive string states of maximal angular momentum into two particles. This will be done for the closed (type-IIA or IIB) superstring theory in flat ten-dimensional spacetime. We will consider all cases: when these particles are both massive, when one of them is massless, and when both of them are massless.

A general formula for the decay can be obtained from a one-loop calculation of the mass-shift. The inverse lifetime $\mathcal{T}^{-1}$ of a massive state of mass $M$ is then given by

$$\mathcal{T}^{-1} = \Gamma = \frac{\operatorname{Im}\Delta M^2}{2M}\,, \tag{1.1}$$

where $\Delta M^2$ represent the radiative correction to the mass, to be expressed as a loop expansion. At one loop, it receives a contribution from the two-particle intermediate states, and thus $\Gamma$ is the lifetime for decaying into two particles.

One can obtain the one-loop expression for $\Delta M^2$ starting from the zero and one loop expressions for the four graviton amplitudes. This was derived in [8] and we will briefly review this computation in the next section 2. We get an integral expression in terms of theta functions. [In [8] a computation of the lifetime was also attempted; however the algorithm employed in this first investigation was not sufficiently accurate and we find now a different and more precise result].

One finds that $\Delta M^2$ is formally expressed as a divergent integral of a positive quantity: its imaginary part can be computed by analytic continuation in $M^2$, starting from $M^2 < 0$ where the integral is convergent. By a systematic expansion of the integrand the calculation then reduces to a sum of integrals of the form

$$I(\alpha, \omega) = \int ds\ s^{-\alpha}\ e^{sM^2\omega}\,, \tag{1.2}$$

where each term has a multiplicity depending on $\alpha$ and $\omega$. The imaginary part is computed by a standard formula:

$$\operatorname{Im} I(\alpha, \omega) = \frac{\pi (M^2\omega)^{\alpha-1}}{\Gamma(\alpha)}\,. \tag{1.3}$$

One finds that this is precisely the expression for the one loop mass-shift in $\phi^3$ field theory, and that $\omega$ is determined by the masses $M_1$ and $M_2$ of the decay products, whereas $\alpha$ is related to their orbital angular momentum, which is necessary to compensate the mismatch of $J$,$J_1$ and $J_2$. This is reviewed in section 3.

The difficult task is to compute the multiplicity of $I(\alpha, \omega)$, due to the large multiplicity of the decay products. We have found a very efficient algorithm for doing that, generalizing the well known saddle-point technique for computing the multiplicity (i.e. entropy) of the string states. The generalities are presented in section 4. The computation of the imaginary part is presented in section 5.

In section 6 we make a numerical analysis of our formulae. This analysis is very fast and accurate. We have made in particular a nontrivial check: $\Delta M^2$ can be expressed in two formally equivalent ways in terms of two different theta functions, which lead to a very different rearrangement of terms in the expansion and to different saddle point analyses. We find nevertheless identical results, summarized in a three dimensional plot, see figures 1 and 4.

We find that $\operatorname{Im}(\Delta M^2)$ can be written as a sum over the contributions of different channels, according to the masses $M_1, M_2$ of the particles of the decay product. It turns out that most of es are exponentially suppressed, except for a line in the space $M_1, M_2$,



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where the decay rate has a power-like dependence on the coupling. This is a surprise, since we do not see any special reason why, given two masses in a three-state vertex, there should be a selection rule for the third. It is likely that that feature results from the interplay among the decay products entropy and centrifugal barrier and phase space effects.[1]

The other surprise is that we find that the lifetime of the excited state of maximal angular momentum for decaying into two particles grows with the first power of its mass $M$. This is computed in section 7.

A final remark: at tree level the whole closed string decay can be described as a sequence of successive two-body decays. If the decay products are narrow resonances — particles with a lifetime much larger than the inverse of their mass — then the lifetime of the first decay is the lifetime of the massive state *tout court*, like for instance the lifetime of a $K$-meson is set by its decay into $\pi$-mesons, despite the $\pi$-meson being also unstable.

## 2. The mass shift

Here we review the derivation of the one-loop expression for $\Delta M^2$ given in [8] by factorizing the one-loop four graviton amplitude obtained by [10] in closed superstring theory (type-IIA and type-IIB give the same result):

$$A_1 = R^4 \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \int \prod_{i=1}^{3} \frac{d^2 z_i}{\mathrm{Im}\,\tau}\; e^{-2\sum_{i<j}^{3} k_i . k_j \chi(z_{ij})} \tag{2.1}$$

where

$$\chi(z) = \log \left| \frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} \right|^2 - 2\pi \frac{(\mathrm{Im}\,z)^2}{\mathrm{Im}\,\tau} \tag{2.2}$$

and $R^4$ is a kinematical factor containing the graviton polarizations. We also set $\alpha' M^2 = 4N$, and choose units where $\alpha' = 4$, so that in what follows we can set $M^2 = N$.

The amplitude $A_1$ has a double pole for $S \to N$ ($S = 2k_1.k_2$ ) due to the propagator of a massive string state with $M^2 = N$, produced by the collision of the two incoming gravitons, and another similar propagator coupled to the two outgoing gravitons. The residue of the double pole is proportional to $\Delta M^2$, the (1-loop) mass shift of the massive state, $M^2 \to M^2 + \Delta M^2$. One has

$$A_1 \to G_{in} \frac{1}{S-N} \Delta M^2 \frac{1}{S-N} G_{out}\,.$$

To get $\Delta M^2$, one divides the double pole residue of $A_1$ by the single pole residue of $A_0$, the tree level four graviton amplitude [10]:

$$A_0 \to G_{in} \frac{1}{S-N} G_{out}\,.$$

The poles of $A_1$ occur as singularities of the integrand in (2.1) for $z_{12}, z_{34} \to 0$. The integrand behaves as $\sim |z_{12}|^{2S} |z_{34}|^{2S} \cdot F(z_{12}, z_{34}, z)$, with $z = \frac{1}{2}(z_1 + z_2)$. One has to look at the terms in the expansion of $F$ that behave as $|z_{12}|^{2N-2} |z_{34}|^{2N-2}$. Further, in order to

[1] We thank Daniele Amati for a discussion on that point.



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select the state of maximal angular momentum $J = 2N+2$, one looks for the maximal power of $\cos\theta$ — the angle between the space-momenta of the ingoing and outgoing gravitons in the c.m. frame – both in the residues of $A_1$ and $A_0$. In this way one obtains

$$\Delta M^2 = c \int \frac{dsd\tau_1}{s^2} \int \frac{d^2z}{s} e^{-\frac{4N\pi^2y^2}{s}} \left| \frac{\pi\theta_1(z|\tau)}{\theta_1'(0|\tau)} \right|^{4N} s^{-2N} \times$$
$$\times \sum_{l=0}^{N-1} \frac{N!^2}{l!^2(N-l-1)!^2} \left| \frac{s}{\pi^2}\partial_z^2 \log\theta_1(z|\tau) + 1 \right|^{2l}, \tag{2.3}$$

where $s = \pi\,\mathrm{Im}(\tau)$ and $c$ is a numerical constant, independent of $N$. In [8] the normalization was checked by evaluating the contribution to $\mathrm{Im}\,\Delta M^2$ from the decay rate of the excited state into two massless states, and finding agreement with the explicit computation of the decay into two gravitons.



## 3. Field theory

Let us first consider the case of $\phi^3$ field theory. Consider the one-loop correction to the propagator of a particle of mass $M$, due to the contribution of particles of masses $M_1$, $M_2$ running in the loop. With a convenient parametrization, the Feynman diagram has the following form

$$\Delta M^2 = \sum_{M_1,M_2,l_0} P \int_0^\infty ds s^{-\beta(l_0)} \int_0^1 d\eta e^{4Ws}, \tag{3.1}$$

with

$$W(\eta) = M^2\eta(1-\eta) - \eta M_1^2 - (1-\eta)M_2^2 .$$

Here $\beta(l_0) = D/2 - 1 + l_0$ ($D$ = spacetime dimension), and the polynomial $P = P(M, M_1, M_2, l_0)$ takes into account in particular the multiplicity of the decay products, $l_0$ being related to their orbital angular momentum needed for matching the angular momentum of the decaying state.

The IR region is $t = \infty$. This integral is convergent below the threshold for particle production. Above the threshold, the integral is defined as usual by analytic continuation, which gives rise to an imaginary part.

The threshold appears when $W(\eta)$ changes sign and becomes positive. The maximum of $W(\eta)$ is at $\eta = \eta_0$, with

$$\eta_0 = \frac{M^2 - M_1^2 + M_2^2}{2M^2},$$

where

$$W(\eta_0) = \frac{1}{4M^2}\left(M^2 - (M_1+M_2)^2\right)\left(M^2 - (M_1-M_2)^2\right) = \vec{p}^{\,2} .$$

Hence $W(\eta_0) > 0$ for $M^2 > (M_1+M_2)^2$.

For future use we define

$$\omega \equiv \frac{4W(\eta_0)}{M^2} = 1 - 2(\sigma_1+\sigma_2) + (\sigma_1-\sigma_2)^2, \qquad \sigma_{1,2} = \frac{M_{1,2}^2}{M^2}. \tag{3.2}$$

Since $4W = M^2\omega - 4M^2(\eta - \eta_0)^2$, for large $M^2$, we can evaluate the integral over $\eta$ by performing a gaussian integration around the maximum at $\eta = \eta_0$. Ignoring constant factors, we get

$$\int_0^1 d\eta\, e^{4Ws} \sim \frac{1}{\sqrt{M^2 s}} e^{s\omega M^2} \,. \tag{3.3}$$

We then evaluate the imaginary part of $\Delta M^2$ by analytic continuation from $M^2\omega < 0$ to $M^2\omega > 0$ by means of the formula seen in eqs. (1.2) and (1.3). The final result is symmetric in $M_1 \leftrightarrow M_2$.

In the case of string theory, we will obtain $\Delta M^2$ expressed in two different (but equivalent) ways, as discussed in section 4. For comparison with the string theory expression studied in detail in appendix A, we set $M^2 = N$ and make the change of variable $\eta = \frac{\pi}{s} y$ to get

$$\Delta M^2 = \sum_{M_1, M_2, l_0} P \int_0^\infty ds\; s^{-\beta(l_0)-1} e^{-4sM_2^2} \int_0^{\frac{s}{\pi}} dy e^{-\frac{4N\pi^2 y^2}{s} + 4\pi y\left(N - M_1^2 + M_2^2\right)} \,. \tag{3.4}$$

In order to compare with the string theory expression studied in section 5, we make another change of variable: $\eta = \frac{\pi}{s} y + \frac{1}{2}$, getting the field theory expression

$$\begin{aligned} \Delta M^2 = & \sum_{M_1, M_2, l_0} P \int_0^\infty ds\; s^{-\beta(l_0)-1} e^{[N - 2(M_1^2 + M_2^2)]s} \times \\ & \times \int_{-\frac{s}{2\pi}}^{\frac{s}{2\pi}} dy\; e^{-\frac{4N\pi^2 y^2}{s} - 4\pi y (M_1^2 - M_2^2)} \,. \end{aligned} \tag{3.5}$$

## 4. General method

Consider the formula for the one-loop string diagram eq. (2.3), which is expressed in terms of $\theta_1(z|\tau)$.

We note that the integrand can be expanded in a sum of terms of the form

$$T(m_1, m_2) = s^{-2N + (m_1 + m_2) - 3} e^{-\frac{4N\pi^2 y^2}{s}} \cdot Q_{m_1} \cdot \bar{Q}_{m_2} \tag{4.1}$$

where

$$Q_m = \left( \frac{\pi \theta_1(z|\tau)}{\theta_1'(0|\tau)} \right)^{2N} \left( \frac{1}{\pi^2} \partial_z^2 \log \theta_1(z|\tau) \right)^m \tag{4.2}$$

can be further expanded in powers of $q^2 = e^{i2\pi\tau}$ and in a Laurent series in $p = e^{2i\pi z}$, which is symmetric under $p \to p^{-1}$. After the integration over $\mathrm{Re}(\tau)$ and $\mathrm{Re}(z)$ (ensuring $L_0 = \bar{L}_0$ on the states), we get a sum of terms like

$$s^{-2N + (m_1 + m_2) - 3} e^{-\frac{4N\pi^2 y^2}{s}} e^{-4\tilde{k}s} e^{4\tilde{j}\pi y} \,, \tag{4.3}$$

with $y = \mathrm{Im}\, z$, $s = \pi\, \mathrm{Im}(\tau)$.

By comparing with eq. (3.4) we see that $2N - (m_1 + m_2) - 2$ corresponds to $l_0$, $\tilde{k}$ to $M_2^2$, and $\tilde{j}$ to $M^2 - M_1^2 + M_2^2$. We can thus determine $P(M, M_1, M_2, l_0)$.



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We will actually compute $\mathrm{Im}\,\Delta M^2$ at fixed $M_{1,2}$ summing over $l_0$, that is summing over every possible angular momentum and multiplicity of the decay products.

It is also useful to make a shift $z \to z + \tau/2$ and to re-express the string one-loop diagram in the form

$$\Delta M^2 = c \int \frac{ds d\tau_1}{s^2} \int \frac{d^2 z}{s} e^{-\frac{4N\pi^2 y^2}{s}} e^{Ns} \left| \frac{\pi \theta_4(z|\tau)}{e^{-i\pi\tau/4}\theta_1'(0|\tau)} \right|^{4N} s^{-2N} \times \\ \times \sum_{l=0}^{N-1} \frac{N!^2}{l!^2 (N-l-1)!^2} \left| \frac{s}{\pi^2} \partial_z^2 \log \theta_4(z|\tau) + 1 \right|^{2l} . \tag{4.4}$$

Now we can expand the integrand of eq. (4.4) in a sum of terms of the form

$$T'(m_1, m_2) = 4^{-2N} s^{-2N+(m_1+m_2)-3} e^{Ns} e^{-\frac{4N\pi^2 y^2}{s}} \cdot Q'_{m_1} \cdot \bar{Q}'_{m_2} , \tag{4.5}$$

where now:

$$Q'_m = \left( \frac{2\pi\theta_4(z|\tau)}{q^{-1/4}\theta_1'(0|\tau)} \right)^{2N} \left( \frac{1}{\pi^2} \partial_z^2 \log \theta_4(z|\tau) \right)^m . \tag{4.6}$$

Note that the new $Q'_m$ has integer power expansions in $q$ and $p$. By looking at the terms $|q|^{2k} = e^{-2ks}$ and $|p|^{2j} = e^{-4j\pi y}$ (after integrating over $\mathrm{Re}(\tau)$ and $\mathrm{Re}(z)$) and comparing with eq. (3.5) we now identify $k = M_1^2 + M_2^2$ and $j = M_1^2 - M_2^2$.

The two forms of $\Delta M^2$, eq. (2.3) and eq. (4.4) are equivalent. However, the final expressions that we will obtain using as starting points eq. (2.3) and eq. (4.4) and computing integrals by saddle-point approximation will involve very different expansions. Therefore it will be a nontrivial check to verify that indeed one gets the same result.

## 5. Calculation of the imaginary part of $M^2$

Here we evaluate $\mathrm{Im}\,\Delta M^2$ using the string loop expression (4.4), written in terms of $\theta_4$.

It should be remembered that the result for $\Delta M^2$ does not change if we replace $\theta_4$ by $\theta_1$, $\theta_2$ or $\theta_3$, since they differ by a shift of $z$ and a factor that compensate the change from $e^{-\frac{4N\pi^2 y^2}{s}}$. In appendix A we repeat the analysis with $\Delta M^2$ expressed in terms of $\theta_1$, eq. (2.3).

We expand the binomial $(\frac{s}{\pi^2}\partial_z^2 \log \theta_4(z|\tau) + 1)^l$ and using the formula

$$\sum_{l=\max(m_1,m_2)}^{N-1} \frac{1}{(l-m_1)!(l-m_2)!(N-l-1)!^2} = \frac{(2N-m_1-m_2-2)!}{(N-m_1-1)!^2 (N-m_2-1)!^2}$$

we get (with $\tau = \tau_1 + i\, s/\pi$ and $z = x + iy$)

$$\Delta M^2 = c \int ds d\tau_1 \int dy dx \times \\ \times \sum_{m_1, m_2} T'(m_1, m_2) \frac{(2N-m_1-m_2-2)! N!^2}{m_1!(N-m_1-1)!^2 m_2!(N-m_2-1)!^2} \tag{5.1}$$

with $T'(m_1, m_2)$ expressed in terms of $Q'_{m_{1,2}}$ as in eq. (4.5). Now we expand

$$Q'_m = \sum_{k,j} c_{kj}(N,m) q^k p^j \,, \tag{5.2}$$

with

$$c_{kj}(N,m) = -\frac{1}{(2\pi)^2} \oint \frac{dq}{q} \oint \frac{dp}{p} q^{-k} p^{-j} Q'_m \,, \tag{5.3}$$

and similarly for the complex conjugate. From the explicit expressions for the $\theta_4$ function given in section 6, one can see that the sum over $k$ contains only positive integer values of $k$, whereas the sum over $j$ contains both positive and negative powers of $j$, with the property $c_{kj} = c_{k(-j)}$.

Let us now consider the integrals over $\tau_1$ and $x$. Since the imaginary part of $\Delta M^2$ comes from the divergence of the integral at $s \to \infty$, we can replace the integral over the fundamental domain by an integral over the full strip. In addition, we note that $\theta_4 \to \theta_3$ by a shift $\tau_1 \to \tau_1 + 1$. Given that the original integral gives the same result for $\theta_3$ and $\theta_4$, we can extend the integration region in $\tau_1$ to the interval $(-1, 1)$. Then

$$\begin{aligned} \frac{1}{2}\int_{-1}^{1} d\tau_1 \int_0^1 dx Q'_{m_1} \bar{Q}'_{m_2} &= \sum_{k,j} (q\bar{q})^k (p\bar{p})^j c_{kj}(N,m_1) \bar{c}_{kj}(N,m_2) \\ &= \sum_{k,j} e^{-2ks-4\pi jy} c_{kj}(N,m_1) \bar{c}_{kj}(N,m_2) \,. \end{aligned}$$

The integration over $y$ is performed by saddle point as in eq. (3.3). Then we consider the integral over $s$, and use the general formula (1.3) for computing the imaginary part. We obtain

$$\begin{aligned} \mathrm{Im}(\Delta M^2) \sim \frac{4^{-2N}}{\sqrt{N}} \sum_{j,k} (N\omega)^{2N+\frac{3}{2}} \sum_{m_1,m_2=0}^{N-1} \sum_{m_1,m_2=0}^{N-1} \frac{(2N-m_1-m_2-2)!}{\Gamma(2N-m_1-m_2+\frac{5}{2})} \times \\ \times \frac{N!^2 \; c_{kj}(N,m_1) \bar{c}_{kj}(N,m_2) (N\omega)^{-m_1-m_2}}{m_1!(N-m_1-1)!^2 m_2!(N-m_2-1)!^2} \,. \end{aligned} \tag{5.4}$$

Here

$$\omega(\rho,\sigma) \equiv 1 - 2\sigma + \rho^2 \,,$$

and

$$\sigma \equiv \frac{k}{N} \,, \qquad \rho \equiv \frac{j}{N} \,. \tag{5.5}$$

We remind that by comparing with field theory we see that the integers $k, j$ are related to the masses $M_1, M_2$ of the decay product (cf. eq. (3.2) ):

$$k = M_1^2 + M_2^2 \,, \qquad j = M_1^2 - M_2^2 \,. \tag{5.6}$$

Here we consider large values of $N$ with fixed $\sigma$ and $\rho$. Other cases will be discussed in appendix B and C.

It will be clear from the calculation below that the main contribution in the sum over $m_1, m_2$ comes from the region where $2N - m_1 - m_2$ is large. Therefore we can approximate

$$\frac{(2N - m_1 - m_2 - 2)!}{\Gamma(2N - m_1 - m_2 + \frac{5}{2})} \sim N^{-7/2}\left(2 - \frac{m_1}{N} - \frac{m_2}{N}\right)^{-7/2} . \tag{5.7}$$

Moreover, in the large-$N$ limit, the sum over $m_{1,2}$ is dominated by a sharp maximum; away from the maximum the terms are exponentially suppressed like $e^{-cN}$. We will see that on the maximum $r(\rho,\sigma) \equiv (2 - (m_1 + m_2)/N)^{-7/2}$ is a finite function of $j/N, k/N$.

Therefore, we can write

$$\mathrm{Im}\left(\Delta M^2\right) \sim 4^{-2N} N^{2N-1/2} \sum_{j,k} \omega^{2N+\frac{3}{2}} r(\rho,\sigma) |L(j,k)|^2 , \tag{5.8}$$

where

$$L(j,k) = \oint \frac{dq}{q} \oint \frac{dp}{p} q^{-k} p^{-j} \left(\frac{2\pi\theta_4(z|\tau)}{q^{-1/4}\theta_1'(0|\tau)}\right)^{2N} H(q,p;j,k) , \tag{5.9}$$

and in turn

$$H(q,p;j,k) = \sum_{m=0}^{N-1} \frac{(N-1)!}{m!(N-m-1)!^2} \left(\frac{\frac{1}{\pi^2}\partial_z^2 \log\theta_4(z|\tau)}{N\omega}\right)^m . \tag{5.10}$$

## 6. Numerical evaluation of $\mathrm{Im}(\Delta M^2)$

We now determine the functions appearing in (5.8). We will make use of the formulas in terms of $q = e^{i\pi\tau_1 - s}$ and $p = e^{i2\pi z}$:

$$\begin{aligned}\frac{2\pi\theta_4(z|\tau)}{q^{-1/4}\theta_1'(0|\tau)} &= \prod_{n=1}^{\infty} \frac{(1 - pq^{2n-1})(1 - p^{-1}q^{2n-1})}{(1 - q^{2n})^2} \\ &\equiv \exp\left[\frac{1}{2} f(q,p)\right] . \end{aligned} \tag{6.1}$$

By expanding the logarithms in the above definition of $f(q,p)$, and interchanging the two infinite sums, we obtain

$$f(q,p) = -2\sum_{n=1}^{\infty} \frac{q^n(p^n + p^{-n} - 2q^n)}{n(1 - q^{2n})} . \tag{6.2}$$

Further:

$$\frac{1}{\pi^2}\partial_z^2 \log\theta_4(z|\tau) = 4\sum_{n=1}^{\infty} \frac{q^{2n-1}[(p + p^{-1})(1 + q^{4n-2}) - 4q^{2n-1}]}{(1 - pq^{2n-1})^2(1 - p^{-1}q^{2n-1})^2} \equiv g(q,p) . \tag{6.3}$$

Define

$$v \equiv \frac{g(q,p)}{\omega(\rho,\sigma)} .$$

Now we use the formula (5.10) for $H = H(q,p,j,k)$:

$$H = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!^2} \left(\frac{v}{n+1}\right)^m = \left(\frac{v}{n+1}\right)^n \frac{1}{2\pi i} \oint \frac{dt}{t^{n+1}} e^{(n+1)t} \left(t + \frac{1}{v}\right)^n , \tag{6.4}$$

with $n = N - 1$. For large-$N$, the integral over $t$ can be computed by a saddle point evaluation. We get

$$H(v,N) \cong N^{-N+1/2} \exp\left[N h(v)\right], \tag{6.5}$$

where

$$h(v) = \log v + \log \frac{\sqrt{1+4v}+1}{\sqrt{1+4v}-1} + \frac{1}{2v}\left(\sqrt{1+4v}-1\right). \tag{6.6}$$

We have checked that this formula provides a very accurate representation for the sum in (5.10) already for $N$ larger than 10.

Thus we finally get

$$L(j,k) = N^{-N+1/2} \oint \frac{dq}{q} \oint \frac{dp}{p} q^{-k} p^{-j} e^{N\left[f(q,p)+h(v(q,p))\right]}. \tag{6.7}$$

The remaining integrals over $q$ and $p$ can also be computed by a saddle-point evaluation. Since the functions appearing in the integrand are complicated, it is more convenient to perform this calculation by a numerical evaluation.

The saddle-point evaluation of the integrals over $q, p$ is done numerically by first finding the extremum of the exponent in eq. (6.7). This determines

$$q_0 = q_0(\rho,\sigma)\,, \qquad p_0 = p_0(\rho,\sigma)\,.$$

We find numerically that the saddle point is obtained for $q_0$ and $p_0$ real and positive. By performing the gaussian integration around the saddle point we get

$$|L(j,k)|^2 \sim N^{-2N+1-2} e^{2NS_L(\rho,\sigma)}, \tag{6.8}$$

where

$$S_L(\rho,\sigma) = -\sigma \log q_0 - \rho \log p_0 + \mathrm{Re}[f(q_0,p_0) + h(v(q_0,p_0))]. \tag{6.9}$$

It is seen that the saddle is a minimum of $S_L$ for $q_0, p_0$ real, and that on it $f$ and $h$ are real.

Finally, from eq. (5.8) we obtain

$$\mathrm{Im}\left(\Delta M^2\right) \sim N^{-3/2} \sum_{j,k} \omega^{3/2}(\rho,\sigma)\; r(\rho,\sigma) e^{2NS_0} \tag{6.10}$$

where

$$S_0(\rho,\sigma) = S_L(\rho,\sigma) + \log\omega(\rho,\sigma) - \log 4\,. \tag{6.11}$$

Also, we mention that the result for (6.6) can also be obtained by evaluating the sum over $m$ in eq. (5.10) by looking at the maximum, found for $m_0 = N(1 - \frac{1}{2v}(\sqrt{1+4v}-1))$, and expanding around it (this is done in the appendix C).

In particular, on the maximum $(2 - \frac{m_1}{N} - \frac{m_2}{N}) = \frac{1}{v}(\sqrt{1+4v}-1)$, since the same maximum holds for $m_{1,2}$. Therefore we find

$$r(\rho,\sigma) = \left(\frac{1}{v(q_0,p_0)}(\sqrt{1+4v(q_0,p_0)}-1)\right)^{-7/2}. \tag{6.12}$$

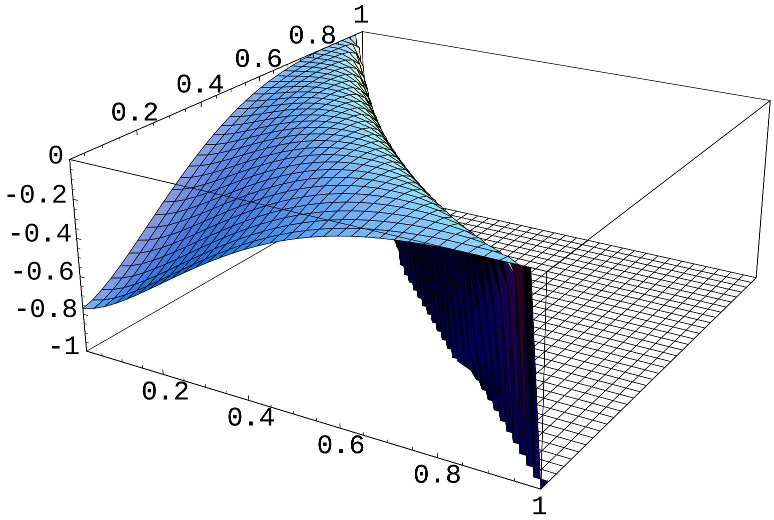


**Figure 1:** The exponent $S_0$ of the decay rate $\Gamma(M_1, M_2)$ in terms of the masses $M_1^2/\sqrt{N}$, $M_2/\sqrt{N}$, computed using the formula (6.11). The kinematically allowed region is inside the triangle defined by $M_1, M_2 > 0$, $\frac{M_1}{M} + \frac{M_2}{M} < 1$. The maximum of $S_0$ is $S_0 = 0$ and it is located on a curve shown in figure 3.

Our final formula for the rate of the decay channel to particles of masses $M_1, M_2$ is thus given by

$$\Gamma(M_1, M_2) = \frac{1}{2\sqrt{N}} \operatorname{Im} \Delta M^2 \bigg|_{M_1, M_2} = \omega^{3/2}\; r \frac{1}{N^2} e^{2NS_0}\,, \tag{6.13}$$

where $S_0$ is a function of the ratios $\frac{M_1}{M}$, $\frac{M_2}{M}$, with $M = \sqrt{N}$.

Note that $\Gamma(M_1, M_2)$ represents the contributions of all decay channels involving particles with the same masses $M_1, M_2$ (since the multiplicity grows exponentially with the mass, for large $M_1, M_2$, there is an exponentially large number of particles contributing to $\Gamma(M_1, M_2)$).

Figure 1 is a numerical plot of $S_0$ in function of $M_1$ and $M_2$. We see that $S_0$ is negative definite, except on some curve (see figure 3) where it identically vanishes. Thus the first observation is that the rate for the decay channel to particles of *generic* masses $M_1$, $M_2$ is exponentially suppressed at large-$N$.

In the appendix A, figure 4 shows the numerical plot of $\tilde{S}_0$, obtained by starting with the expression (2.3) in terms of $\theta_1$. The two figures are identical.

Figure 2 is a plot of $S_0(M_1)$ for given $M_2$, i.e. slices of figure 1 at constant $M_2$. One can see that the maximum exactly passes by $S_0 = 0$. This happens for any $M_2$. We have checked that the factor $\omega^{3/2}(\rho, \sigma)\, r(\rho, \sigma)$ in eqs. (6.10) and (6.13) is finite inside the allowed triangle, except on the boundary $M_1 + M_2 = M$ where $\Gamma$ is anyhow suppressed.

Modulo the exponentially suppressed processes, a massive particle will decay through the special channel where $S_0$ vanishes. This defines a curve $M_2 = F(M_1)$ in the space $M_1, M_2$, which is shown in figure 3. Such dominant channels exhibit a power-like behavior

$$\Gamma(M_1, M_2) \sim N^{-2}\,. \tag{6.14}$$

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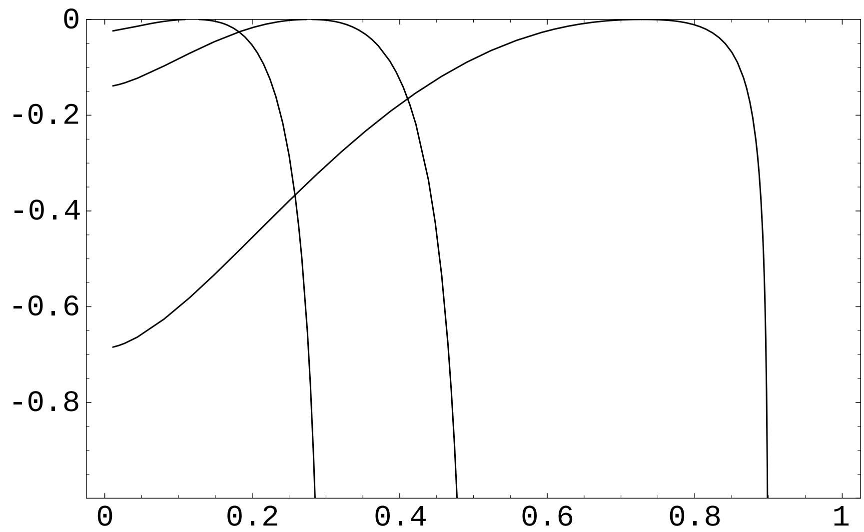


**Figure 2:** Sections of figure 1 at constant $M_2$. The three curves displayed corespond to $\frac{M_2}{M} = 0.1, 0.3, 0.7$ (with maxima located at $\frac{M_1}{M} \sim 0.74,\ 0.28,\ 0.12$ respectively). One can see that the maximum always passes by $S_0 = 0$ for every value of $M_2$.

It is remarkable that for large-$N$ the dynamics "excludes" decays into kinematically allowed channels. In other words, we find that if the massive particle decays into a particle of mass $M_1$, the mass of the other particle $M_2$ is uniquely determined, modulo exponentially suppressed processes.

The curve $M_2 = F(M_1)$ is well approximated by the curve

$$\left(\frac{M_1}{M}\right)^a + \left(\frac{M_2}{M}\right)^a = 1\,, \qquad a \cong 0.73\,, \tag{6.15}$$

also shown in figure 3. Although this not the true analytical formula connecting $M_1, M_2$ (which is extremely complicated), eq. (6.15) is useful as a book-keeping of the approximate relation between $M_1$ and $M_2$.

In the appendices B and C we have considered the cases when one of the masses $M_1$ or $M_2$, or both, are small with respect to $N$. We find that the only case which is not exponentially suppressed is when $M_2 = 0$ and $M_1^2 = N - j$ with $j$ finite (or *viceversa*). In this case

$$\Gamma \sim N^{-5/2}\,.$$

Thus the dominant channel is for both $M_{1,2}^2$ of order $N$ along the curve $M_2 = F(M_1)$, where the decay width is given by eq. (6.14).

## 7. Lifetime of a state with maximum angular momentum

Writing the formula (6.10) for $\mathrm{Im}(\Delta M^2)$ in terms of $\sigma_{1,2} = M_{1,2}^2/N$, we obtain

$$\mathrm{Im}\left(\Delta M^2\right) = N^2 \int_{\mathcal{D}} d\sigma_1 d\sigma_2 \frac{1}{N^{3/2}}\ e^{2NS_0}\,, \tag{7.1}$$

where the domain of integration is the region $\sigma_1, \sigma_2 > 0$ and $\sqrt{\sigma_1} + \sqrt{\sigma_2} < 1$.

We have seen that $S_0$ is in general negative except on the curve where $S_0$ vanishes. Thus only a small neighborhood of this curve contributes to the integral (7.1). Let $l$ be a parameter along the line, $0 < l < 1$, and let $n$ a parameter for the orthogonal direction, where $n = 0$ means a point on the curve. It is convenient to use $n$, $l$ as integration variables. The integration over $l$ is trivial, since $S_0$ takes the same value (i.e. equal to zero) for all $l$. In the vicinity of the line, we can expand $S_0$ in powers of $n$, and get a gaussian integral over $n$ of the form

$$\mathrm{Im}\left(\Delta M^2\right) = N^2 \int dn \frac{1}{N^{3/2}} e^{-cNn^2} \,, \tag{7.2}$$

where $c$ is a number of order 1. The gaussian integral gives an extra factor $N^{-1/2}$, so we get (see eq. (1.1) )

$$\mathcal{T}^{-1} = \frac{\mathrm{Im}\left(\Delta M^2\right)}{2\sqrt{N}} = \mathrm{const.} \frac{1}{\sqrt{N}} \tag{7.3}$$

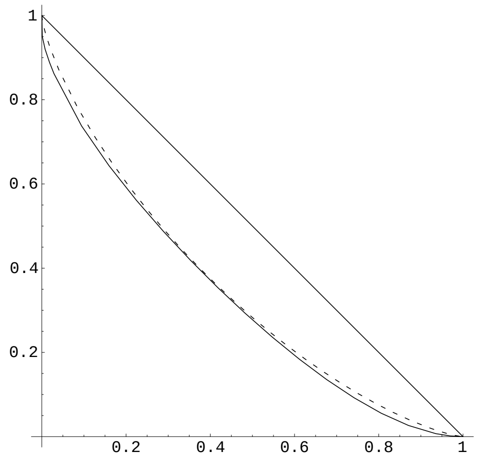


**Figure 3:** The curve defined by $S_0(M_1, M_2) = 0$ (solid line), describing the dominant channels. For these special values of $M_1, M_2$ the rate is not exponentially suppressed. The dashed line is the curve $(M_1/M)^a + (M_2/M)^a = 1$, $a = 0.73$.

or

$$\mathcal{T} = \mathrm{const}\alpha' M \,. \tag{7.4}$$

where we have restored $\alpha'$. Thus the lifetime of a state with maximum angular momentum in closed superstring theory is proportional to the mass.

## Acknowledgments

We would like to thank D. Amati for a useful discussion. R.I. acknowledges partial support by the European network HPRN-CT-2000-00131. J.R. wishes to thank the Abdus Salam International Centre for Theoretical Physics for hospitality while part of this work was carried out.

## A. Alternative calculation of $\mathrm{Im}(\Delta M^2)$

Here we provide an alternative calculation of $\mathrm{Im}(\Delta M^2)$ by using as starting point the formula (2.3) in terms of $\theta_1$.

The computation is quite similar to the one done in sections 5 and 6, this time using the expansion of eqs. (4.1) and (4.2). We also make use of the formulas

$$\begin{aligned} \frac{\pi\theta_1(z|\tau)}{\theta_1'(0|\tau)} &= \sin\pi z \prod_{n=1}^{\infty} \frac{(1-pq^{2n})(1-p^{-1}q^{2n})}{(1-q^{2n})^2} \\ &\equiv \exp\left[\frac{1}{2}\tilde{f}(q,p)\right] \end{aligned} \tag{A.1}$$

$$\frac{1}{\pi^2}\partial_z^2 \log\theta_1(z|\tau) = \frac{4}{p+p^{-1}-2} + 4\sum_{n=1}^{\infty} \frac{q^{2n}\left[\left(p+p^{-1}\right)\left(1+q^{4n}\right)-4q^{2n}\right]}{\left(1-pq^{2n}\right)^2\left(1-p^{-1}q^{2n}\right)^2}$$
$$\equiv \tilde{g}(q,p)\,. \tag{A.2}$$

By expanding the logarithms in the above definition of $\tilde{f}(q,p)$, and interchanging the two infinite sums, we now obtain

$$\tilde{f}(q,p) = \log\left[\frac{1}{4}\left(p+p^{-1}-2\right)\right] - 2\sum_{n=1}^{\infty}\frac{q^{2n}(p^n+p^{-n}-2)}{n(1-q^{2n})}\,. \tag{A.3}$$

Note that $\tilde{f}$ can be defined modulo $i\pi$, that is modulo the sign inside the logarithm, since in eq. (4.4) only $\exp\left(2\,\mathrm{Re}\,\tilde{f}\right)$ appears.

Both $\tilde{f}$ and $\tilde{g}$ are even functions of $q$. We get the same formulas (5.9) , (5.10), (6.5) and (6.6), with $f\to\tilde{f}$ and $g\to\tilde{g}$ and thus $v\to\tilde{v}$. Moreover, as explained in section 4, the relation with the masses $M_{1,2}$ of the exponents $\tilde{j},\tilde{k}$ of the expansions $q^{2\tilde{k}}$, $p^{\tilde{j}}$ is now different. In terms of the exponents $j,k$ of section 5 we have

$$\tilde{k} = \frac{(k-j)}{2}\,, \qquad \tilde{j} = M^2 - j\,.$$

In conclusion we obtain

$$\mathrm{Im}\left(\Delta M^2\right) \sim N^{-3/2}\sum_{j,k}\omega^{3/2}(\rho,\sigma)\tilde{r}(\rho,\sigma)e^{2N\tilde{S}_0(\rho,\sigma)}\,, \tag{A.4}$$

where

$$\tilde{S}_0(\rho,\sigma) = -(\sigma-\rho)\log\tilde{q}_0 - (1-\rho)\log\tilde{p}_0 + \mathrm{Re}\left[\tilde{f}(\tilde{q}_0,\tilde{p}_0) + h(\tilde{v}(\tilde{q}_0,\tilde{p}_0))\right] + \log\omega(\rho,\sigma)\,, \tag{A.5}$$

$$\sigma \equiv \frac{M_1^2+M_2^2}{N}\,, \qquad \rho \equiv \frac{M_1^2-M_2^2}{N}\,,$$

and the saddle point values $\tilde{q}_0,\tilde{p}_0$ correspond to the stationary point of $\tilde{S}_0$.

The functions that enter into the final expression (A.5) are individually very different from the ones appearing in the formula for $S_0$ in section 6. The result is however the same, to a surprising degree of accuracy, see figure 4.

## B. Decay into two massless particles

The decay rate of the massive particle into two massless particles (e.g. gravitons) can also be obtained from the general formula (4.4). We need to consider the special channel with $M_1 = M_2 = 0$, i.e. the powers $q^k p^j$ in the expansion with $k = j = 0$. This is obtained by formally setting $g(q,p)\to 0$ (since it starts with $q^1$) and $\frac{2\pi\theta_4(z|\tau)}{q^{-1/4}\theta_1'(0|\tau)}\to 1$. For this $k = j = 0$ term, we thus get

$$\Delta M^2\Big|_{\text{massless}} = c\int\frac{ds\,d\tau_1}{s^2}\int\frac{d^2z}{s}e^{-\frac{4N\pi^2y^2}{s}}e^{Ns}\left(\frac{1}{4s}\right)^{2N}\frac{N^2(2N-2)!}{(N-1)!^2}\,. \tag{B.1}$$

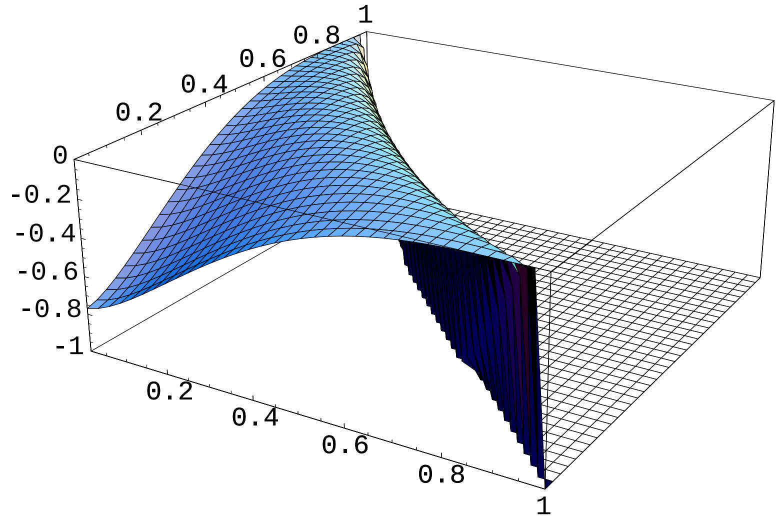


**Figure 4:** The exponent $\tilde{S}_0$ of the decay rate $\Gamma(M_1, M_2)$ in terms of the masses $M_1/M$, $M_2/M$, computed using the formula (A.5). The plot should be compared to figure 1, computed by following a different way.

Computing the integrals over $x, y$ and $\tau_1$, we are left with an integral over $s$ whose imaginary part gives

$$\mathrm{Im} \int ds s^{-\frac{5}{2}-2N} e^{Ns} \cong \frac{1}{\sqrt{N}} e^{2N} 2^{-2N}$$

Hence

$$\mathrm{Im}\, \Delta M^2 \Big|_{\text{massless}} \sim \sqrt{N} e^{-2N(\log 4 - 1)} . \tag{B.2}$$

This is exponentially decreasing for large-$N$.

Note that this decay corresponds to the corner $M_1 = M_2 = 0$ in figure 1, which is also exponentially decreasing. Indeed, the numerical value at $M_1 = M_2 = 0$ of the plot is $\cong -0.7726$, which agrees with $-2(\log 4 - 1)$.

In [8] this result has been compared with the explicit direct computation of the decay into two gravitons, finding complete agreement.

## C. Decay into a massless and a massive particle, and remaining cases

Another interesting special case is when the massive particle decays into a massless particle (e.g. graviton) and a massive particle. So, consider the decay $M \to M_1 + M_2$ with $M_2 = 0$ and $M_1^2 = N - j$ (we remind that $M^2 = N$). We consider both the case when $j$ is a finite integer and the case $j = c\, N$ where $c$ can be a constant less than one or $c \sim N^{\lambda - 1}$ with $\lambda < 1$. The case $M_2 = 0$ is most suitably treated by the formula (2.3) expressed in terms of $\theta_1(z|\tau)$, setting $q = 0$ to isolate the term corresponding to $M_2 = 0$ in the expression (4.3). Thus (see (A.1), (A.2) )

$$\frac{\pi \theta_1(z|\tau)}{\theta_1'(0|\tau)} \to \sin(\pi z) , \qquad \frac{1}{\pi^2} \partial_z^2 \log \theta_1(z|\tau) \to -\frac{1}{\sin^2(\pi z)} .$$

Therefore now

$$u(p) \equiv g(q=0,p) = \frac{4}{\left(p^{1/2} - p^{-1/2}\right)^2}\,, \tag{C.1}$$

$$e^{f(q=0,p)} = \left(\frac{\pi\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right)^2 = -u(p)^{-1}\,. \tag{C.2}$$

Now $\omega = (\frac{j}{N})^2$ thus $\omega N = j^2/N$ and $\frac{g}{\omega N} = \frac{u(p)N}{j^2}$. By following similar steps as those which led to eq. (5.4), now we obtain

$$\begin{aligned}\operatorname{Im}\Delta M^2 \sim \frac{(\omega N)^{2N+\frac{3}{2}}}{\sqrt{N}} &\oint \frac{dp}{p} p^{-j}(u(p))^{-N} \oint \frac{d\bar{p}}{\bar{p}} \bar{p}^{-j}(u(\bar{p}))^{-N} \\ &\times \sum_{m_1,m_2=0}^{N-1} \frac{(2N-m_1-m_2-2)!}{\Gamma(2N-m_1-m_2+\frac{5}{2})}\, Z_{m_1}\, \bar{Z}_{m_2}\,.\end{aligned} \tag{C.3}$$

where

$$Z_m \equiv Z(u(p),m,j,N) \equiv \frac{N!}{m!(N-m-1)!^2}\left(\frac{u(p)N}{j^2}\right)^m.$$

We use a saddle point technique to sum over $m_{1,2}$: the dominant contribution comes from the maximum of the exponential dependence in those variables. Using the Stirling formula, we find

$$Z_m \sim (N-m-1)\sqrt{\frac{N}{m}}\, e^{N\log N} e^I\,, \tag{C.4}$$

$$\begin{aligned} I &= N - m - 2N\log(N-m-1) \\ &+ m\left(-\log m + 2\log(N-m-1) + \log\frac{u(p)N}{j^2}\right).\end{aligned} \tag{C.5}$$

Imposing that the derivative in $m$ of the exponent vanishes, we get the equation for $m_0$, the maximum locus,

$$(N-m_0-1)^2 = \frac{m_0 j^2}{u(p)N}\,,$$

which is solved to give

$$N - m_0 = 1 - \frac{j^2}{2Nu} + \sqrt{(1-\frac{j^2}{2Nu})^2 + \frac{j^2}{u} - 1}\,. \tag{C.6}$$

Let us now discuss different cases.

For $j^2/N \to 0$ , eq. (C.6) gives

$$m_0 \cong N - \frac{j}{\sqrt{u}} - 1\,. \tag{C.7}$$

On the maximum, we get

$$Z(u(p),m_0,j,N) \sim \frac{j}{\sqrt{u}}\left(\frac{N}{j^2}\right)^N u^N \exp\left[\frac{j}{\sqrt{u}}\right],$$

and

$$\frac{(2N-m_1-m_2-2)!}{\Gamma\left(2N-m_1-m_2+\frac{5}{2}\right)} \sim j^{-7/2}\left(\frac{1}{\sqrt{u(p)}}+\frac{1}{\sqrt{u(\bar{p})}}\right)^{-7/2} .$$

The second derivative of the exponent $I$ is $-1/m_0 - 2/(N-m_0-1)$, which in this limit is $2\sqrt{u}/j$. We are left with a gaussian integral in $\delta m_1$, $\delta m_2$ with a spread of order $\sqrt{j}$, which is small compared to the range of $m$. Hence

$$\operatorname{Im}\Delta M^2 \sim \frac{(\omega N)^{2N+\frac{3}{2}}}{\sqrt{N}}\left(\frac{N}{j^2}\right)^{2N} j^{-7/2}\left|j^{3/2}\oint\frac{dp}{p}p^{-j}\exp\frac{j}{\sqrt{u}}\right|^2 , \qquad \text{(C.8)}$$

where we have neglected finite powers of $u$ and kept into account the range in $m$. We recall that $1/\sqrt{u} = \frac{1}{2}(p^{1/2}-p^{-1/2})$. The remaining integral over $p$ is obtained again by saddle point technique: defining $x = p^{1/2}$, we require the derivative in $x$ of the exponent to vanish, that is $-2x^{-1}+\frac{1}{2}(1+x^{-2}) = 0$. The solution is $x = x_0 = 2+\sqrt{3}$, discarding the other solution $x = 1/x_0$ which would make $N-m \sim j/u(p) < 0$. We get

$$\oint\frac{dp}{p}p^{-j}\exp\left[\frac{j}{2}\left(p^{1/2}-p^{-1/2}\right)\right] \sim \frac{1}{\sqrt{j}}\exp\left[-c_0 j\right] ,$$

where

$$c_0 = 2\log x_0 - \frac{x_0^2-1}{2x_0} \cong 0.9 .$$

Thus we finally obtain

$$\operatorname{Im}\Delta M^2 \sim \frac{j^{3/2}}{N^2}\exp\left[-2c_0 j\right] . \qquad \text{(C.9)}$$

For large $j$, this rate is exponentially suppressed.

The calculation applies as well to the case of finite $j$. In this case, the exponential factor depending on $j$ is a finite number of order $O(1)$, so one obtains

$$\Gamma \sim N^{-5/2} .$$

Now consider the case when $N > j \geq N^{1/2}$. Then eq. (C.6) gives $N-m_0-1 \sim \frac{j}{u}-\frac{j^2}{2Nu}$, and therefore we find the same result, up to negligible corrections.

Finally, consider the case $j = cN$. Then eq. (C.6) reduces to

$$N-m = \frac{1}{2}\left(\frac{Nc^2}{u}-2\right)\left(\sqrt{1+\frac{4u}{c^2}}-1\right) ,$$

and we have to evaluate

$$\oint\frac{dp}{p}p^{-cN}\exp\left[N\left(-\log u - 2\log\frac{\left(\sqrt{1+\frac{4u}{c^2}}-1\right)c^2}{2u}+\frac{c^2}{2u}\left(\sqrt{1+\frac{4u}{c^2}}-1\right)\right)\right] .$$

We note that this is a particular case of the expression of the general case (6.7) – we recognize $\tilde{v}(q=0,p) = u(p)/c^2$ and the expression $N(\tilde{f}(q=0,p)+h(\tilde{v}(q=0,p))$ in the exponent – except that in the general case $M_2^2 \sim N$ we have an extra gaussian integral coming from the integration over $q$ (rather than setting $q = 0$ to isolate the term corresponding to $M_2^2 = 0$). As a result, (6.7) will give an extra factor $|1/\sqrt{N}|^2$.

Summarizing, for $M_2 = 0$ and $M_1^2 = N - j$ with $1 \leq j \leq cN$, we get

$$\operatorname{Im} \Delta M^2 \sim \frac{j^{3/2}}{N^2} \exp\left[-bj\right] \tag{C.10}$$

where $b > 0$ is a numerical coefficient of order 1.

The case when $j = N$, i.e. $c = 1$, corresponds to the $M_1 = M_2 = 0$ case treated in appendix B. It can also be recovered as follows. We need to compute the coefficient of the power $p^N$ in the Laurent expansion of

$$\exp\left[N\left(-\log u - 2\log\frac{\sqrt{1+4u}-1}{2u} + \frac{1}{2u}\left(\sqrt{1+4u}-1\right)\right)\right] = \left(\frac{p}{4}\right)^N \exp\left[N\left(1 + \sum_{n>0} c_n p^{-n}\right)\right]. \tag{C.11}$$

The coefficient of $p^N$ is $(\frac{e}{4})^N$. Now we have to take the modulus square of this coefficient, and take into account that the generic case eq. (C.10) has an extra factor $|1/\sqrt{j}|^2$ originating from the gaussian integral around the saddle point over $p$. We thus obtain the correct result for $M_1 = M_2 = 0$:

$$\operatorname{Im} \Delta M^2 \sim N^{1/2} \cdot \left(\frac{e}{4}\right)^{2N}, \tag{C.12}$$

in agreement with appendix B.

Finally, we have also considered the case when $M_2^2 = n$ is finite or small with respect to $N$, which is allowed for $j > 2\sqrt{nN} - n$. We find that this case is exponentially suppressed.

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